A Brief Review of Probability Theory

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Simple Probabilities

- Classical probability rule: count all the points in a given sample space and assign them equal probabilities
- Given W points in the sample space, the probability of each point is $1/W$
- $p_i = \frac{1}{W}$
- Example: Two outcomes of a coin toss (H or T)
  $p_H = 1/2$
  $P_T = 1/2$

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Example: coin toss

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- If $N = 1$, 2 outcomes
- $N = 2$, 4 outcomes (HH, TT, HT, TH)
- $N = 3$, 8 outcomes (HHH, HTT, $\cdots$ TTT)
- In general, 2^N outcomes
- Key: each outcome counted separately even though we may not be able to distinguish one "head" from another
- Probability of any *one event*

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Enumeration of Simple Events

1. We have enumerated all possible *simple events* and assign equal probabilities to each
2. *Simple event*: **the outcome of a trial that doesn't depend on any other event**
3. Coin flipping works very well. Other cases?
4. Predicting sex of a baby
   - How many outcomes?
   - 2
   - What is the probability any given birth will be a boy?
   - Should be $p_{boys} = 0.5$ BUT $p_{boys} = 0.51$ for many countries.

What is wrong with our analysis?

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Another Example

- Will it rain today?
- Two possible outcomes (in Rio): rain (R) or shine (S)
- In South Bend, we have rain, shine, snow, and sleet
- Is $p_{rain} = 0.5$? Why not?

Simple vs Compound Events

- The problem: births, weather, and other such things are *not simple events*
- They are *compound events*
- Compound event: A collection of simple events
- Outcome depends on product of many simple event probabilities

Must be careful and test any *a priori* assignment of probabilities we make.

Statistical Probability

- *Statistical probability* assignment of probabilities to events by measuring the relative frequency of occurrence
- Example: Statistical probability of a person's eye color
- How to determine? Go make the measurements!
- Graduate students examine 1000 people, report 601 brown, 251 blue, and 148 green

$$p_{brown} = 0.601,\ p_{blue} = 0.251, and\ p_{green} = 0.148$$

Is this "correct"?

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Statistical Probability Example

As usual, the advisor of these students isn't satisfied. He tells them to go out and make more measurements

- New sample size is 10,000
- Results

$$n_{brown} = 6,205, \; n_{blue} = 2,688, \; n_{green} = 1,107$$

- Revised probabilities

$$p_{brown} = 0.6205, \; p_{blue} = 0.2688, \textit{and } p_{green} = 0.1107$$

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Statistical Probability

- Relative frequencies are (n_i/N), where n_i is the number of occurrences and N are the total number of samples
- If relative frequencies tend toward a constant as N goes to infinity, then this limit is defined as the *statistical probability*, p_i

$$p_i = \lim_{N\to\infty} \left(\frac{n_i}{N}\right) \tag{2}$$

- In practice, (n_i/N) will fluctuate, but should converge
- For random events, fluctuations in statistical probability diminish as $N^{-1/2}$
- Larger N, the more accurate is p_i.

Probability

- Key point 1: statistical probabilities are only accurate when the same event occurs many times, so that the ratio (n_i/N) can converge
- It is incorrect to speak of the "statistical probability" of a team winning a football match, since the contest will only occur once
- Each game is played under different conditions
- Key point 2: Statistical probabilities only have meaning when (n_i/N) tends to a limit
- What is the statistical probability the Dow Jones Industrial Average will close at 10,000 tonight?
- The market is not converging, so there is no "statistical probability" associated with it

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Axioms of Probability Theory

1. All probabilities are either *zero or positive numbers*
   Easy to see from definition: $p_i = n_i/N$
2. All probabilities are *less than or equal to one*
   Also easy to see from the definition
3. For *mutually exclusive* events, the probability of either event i occurring or j occurring is

$$p_{i+j} = p_i + p_j \tag{3}$$

   where the notation p_{i+j} refers to the probability that either event i *or* j occur.

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Distributions

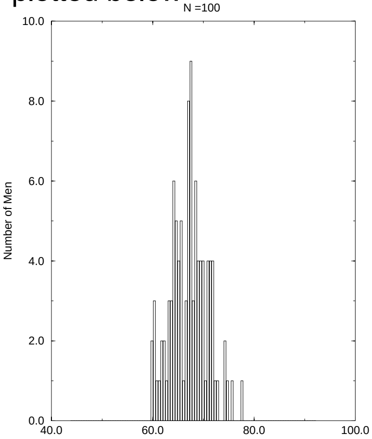
- Let's go back to our team of graduate students
- Now the advisor wants to know the height of the average American adult male
- The students measure heights (in inches, this is the US)
- After measuring 100 men, they compute the average height using

$$\langle h \rangle = \frac{1}{N} \sum_{i=1}^{N} h_i \tag{4}$$

  where $\langle h \rangle$ is the average height, N is the total number of men (samples) measured, and h_i is the result of measurement i.
- They obtain a value of $\langle h \rangle = 67.4$ inches

Height Example

The raw data are plotted below



Height Example, cont.

- The ruler has a resolution of 1/2 in, so heights were *binned*
- Thus, the *actual* formula used for the mean was

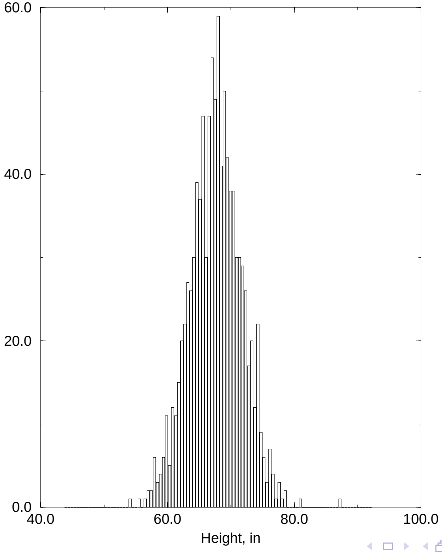
$$\langle h \rangle = \frac{\sum_{i=1}^{nbins} n_i h_i}{N} \tag{5}$$

- $N = \sum_{i=1}^{nbins} n_i$ is the sum over all the bins of the number of men having a height within some discrete bin width
- In the limit of $N \to \infty$, eqn 5 goes to

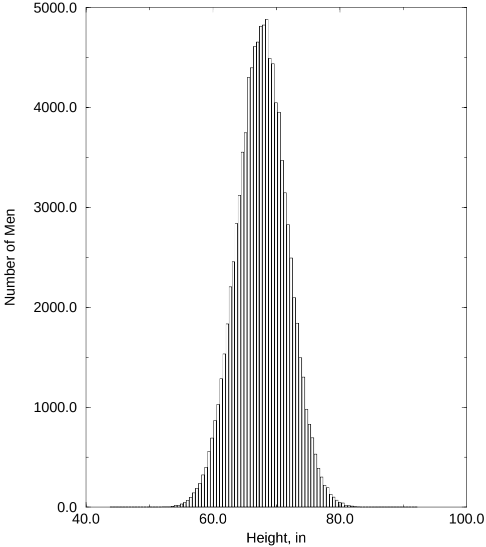
$$\langle h \rangle = \sum_{i=1}^{nbins} p_i h_i \tag{6}$$

- p_i is the *statistical probability* that a man is of height h_i

The graduate students are so excited they take more samples with $N = 1000$



...and $N = 100,000$


5000.0
4000.0
3000.0
2000.0
1000.0
0.0
Number of Men
40.0
60.0
80.0
100.0
Height, in

Distributions

- As the number of samples increases, the distribution becomes smoother and smoother
- Given a ruler with fine resolution dh, $p_i \rightarrow p(h)dh$
- dh is the differential “bin width”
- $p(h)$ is a smooth function of h
- Continuous curve $p(h)$ is the *probability density distribution*
- Sum over bins becomes an integral

$$\sum_{i=1}^{nbins} p_i = 1 \tag{7}$$

becomes

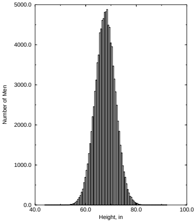
$$\int_{-\infty}^{+\infty} p(h)dh = 1 \tag{8}$$

Continuous Distributions

$$\int_{-\infty}^{+\infty} p(h)dh = 1 \tag{9}$$

We write the above equation in the general case; for the example we were talking about, the lower limit would obviously be zero. Note that $p(h)$ must be a probability *density* distribution with units of $(h)^{-1}$.

Gaussian Distribution



- The distribution which fits the data above is the most important probability distribution in statistical mechanics
- The *Gaussian* or *normal* distribution

$$p(h) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2}\left(\frac{h-<h>}{\sigma}\right)^2\right] \qquad (10)$$

- Two parameters: the mean value ($<h>$) and standard deviation (σ)

Discrete Probability Distributions

- Let $F(x)$ be the value of a discrete function at x
- If there are M possible values of F $(F(x_1), F(x_2), \ldots F(x_M))$ with probabilities P $(P(x_1), P(x_2), \ldots P(x_M))$ then

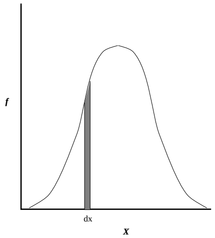
$$\langle F(x) \rangle = \frac{\sum_{j=1}^{M} P(x_j)F(x_j)}{\sum_{j=1}^{M} P(x_j)} \tag{11}$$

- $P(x)$: *discrete distribution*; $F(x)$: *discrete random variable*.
- Since P is a probability, we know it is normalized.

$$\sum_{j=1}^{M} P(x_j) = 1 \tag{12}$$

$$\langle F(x) \rangle = \sum_{j=1}^{M} P(x_j)F(x_j) \tag{13}$$

Continuous Distributions

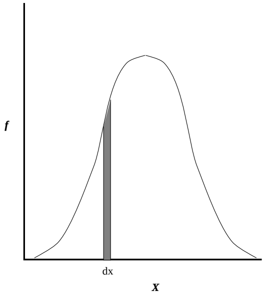


- Let f be a *continuous distribution function* of events depending on x
- Let $\rho\, dx$ be probability of event occurring in region dx

$$\rho\, dx = \frac{f\, dx}{\int_{-\infty}^{+\infty} f\, dx} \tag{14}$$

- ρ is the *probability density*

Continuous Distributions



- Normalized *probabilities* $\int_{-\infty}^{+\infty} \rho \, dx = 1$
- Averages are calculated as follows

$$\langle F \rangle = \frac{\int F \, f(x) \, dx}{\int f(x) \, dx} \tag{15}$$

Gaussian Distribution

We have already encountered a Gaussian distribution. Using the symbols for this section, the Gaussian distribution has the form

$$P(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[\frac{-(x - \langle x \rangle)^2}{2\sigma^2}\right] \tag{16}$$

Summary

- Simple probabilities
- Statistical probability derived from measuring frequency of occurence of eents
- Relative frequencies of events must converge for large number of samples to have a statistical probability
- Discrete and continuous probability distributions
- Weight individual occurences by distribution to get average
- Most important is the Gaussian distribution

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